

Szegő-Widom asymptotics of Chebyshev Polynomials on Circular Arcs

Benjamin Eichinger

July 26, 2016

Abstract

Thiran and Dettaille give an explicit formula for the asymptotics of the sup-norm of the Chebyshev polynomials on a circular arc. We give the so-called Szegő-Widom asymptotics for this domain, i.e., explicit expressions for the asymptotics of the corresponding extremal polynomials. Moreover, we solve a similar problem with respect to the upper envelope of a family of polynomials uniformly bounded on this arc. That is, we give explicit formulas for the asymptotics of the error of approximation as well as of the extremal functions. Our computations show that in the proper normalization the limit of the upper envelope represents the diagonal of a reproducing kernel of a certain Hilbert space of analytic functions. Due to Garabedian the analytic capacity in an arbitrary domain is the diagonal of the corresponding Szegő kernel. We don't know any result of this kind with respect to upper envelopes of polynomials. If this is a general fact or a specific property of the given domain, we rise as an open question.

1 Introduction

Getting explicit asymptotics is a fundamental problem in constructive approximation theory. The problem can be related to both, approximation error and the function of best approximation. Usually, the second problem is essentially harder. For instance, it took almost 100 years that Lubinsky [6] characterized the extremal function in the famous Bernstein problem on the best approximation in $[-1, 1]$ of $|x|^\alpha$ by polynomials. A problem, which is closer related to the one that we are going to consider in this paper was recently solved by Christiansen, Simon and Zinchenko [3] (to appear in Invent. math.). They showed that if $E \subset \mathbb{R}$ is regular, compact and satisfies the Parreau-Widom condition, then the Chebyshev polynomials obey

$\|T_n\|_E \leq Q\text{Cap}(E)^n$, where $\text{Cap}(E)$ is the logarithmic capacity of E and $\|\cdot\|_E$ denotes the sup-norm. In this case $\overline{\mathbb{C}} \setminus E$ may be infinitely connected. Under the restriction that E is a finite union of intervals, they were also able to describe the asymptotics of the extremal functions T_n .

For further references on Chebyshev polynomials and its asymptotics see [9, 10, 12, 13, 14]. We consider Chebyshev polynomials on circular sets A_α , of the form

$$A_\alpha = \{u \in \mathbb{C} : |u| = 1, -\alpha \leq \arg u \leq \alpha\}, \quad 0 < \alpha < \pi.$$

It is motivated by a paper of Thiran and Dettaille [11], who showed that the extremal value obeys

$$\|T_n\|_{A_\alpha} \sim \cot(\alpha/4)\text{Cap}(A_\alpha)^{n+1}. \quad (1.1)$$

Our approach is completely different and allows us to find:

- (i) explicit asymptotics of the Chebyshev polynomials,
- (ii) explicit asymptotics of the upper envelope of the family $\mathcal{P}_{n,\alpha}$ of polynomials of degree at most n which are bounded by one in modulus on A_α ; cf. (2.1).

To be more precise, let $g_\Omega(z, z_0)$ denote the Green's function of the point z_0 and the domain Ω . Writing $i * g_\Omega(z, z_0)$ for the harmonic conjugate of $g_\Omega(z, z_0)$ we define the complex Green's function of the domain by

$$b_\Omega(z, z_0) = e^{-(g_\Omega(z, z_0) + i * g_\Omega(z, z_0))};$$

cf. [14]. Instead of the Chebyshev polynomials, let us consider the normalized polynomials $P_{n,\infty}$, i.e., the polynomial in $\mathcal{P}_{n,\alpha}$ that has maximal leading coefficient. Set $\Omega_\alpha = \overline{\mathbb{C}} \setminus A_\alpha$. Due to Montel's theorem, at least by passing to subsequences, the family $b_{\Omega_\alpha}(u, \infty)^n P_{n,\infty}(u)$ has a limit as $n \rightarrow \infty$. In Theorem 2.4 we present the limit function explicitly. Following the notion introduced in [3], we say that Ω_α has Szegő-Widom asymptotics.

The leading coefficient reflects the behavior of the polynomial at ∞ . It is therefore natural to consider this problem also for other points $u_0 \in \Omega_\alpha$. By $P_{n,u_0} \in \mathcal{P}_{n,\alpha}$ we denote those polynomials in $\mathcal{P}_{n,\alpha}$ which have maximal value at the point u_0 . Due to the symmetry of the domain, it suffices to consider the problem for $|u_0| < 1$; see (2.12).

Let $\lambda : \Omega_\alpha \rightarrow \Pi = \{\lambda \in \mathbb{C} : -\frac{\pi}{4} \leq \arg \lambda \leq \frac{\pi}{4}\}$ be defined by

$$\lambda(u) = \left(\frac{ue^{i\alpha} - 1}{u - e^{i\alpha}} \right)^{1/4}. \quad (1.2)$$

Theorem 1.1. *Let $|u_0| < 1$ and $\lambda_0 = \lambda(u_0)$. There exists $\phi \in \mathbb{R}$ such that*

$$\lim_{n \rightarrow \infty} b_{\Omega_\alpha}(u, \infty)^n P_{n, u_0}(u) = e^{i\phi} \frac{1}{2} \left(1 + \frac{h(\lambda, \lambda_0)}{h(\lambda_0, \lambda_0)} \right) \frac{\lambda^2 - |\lambda_0|^2}{\lambda^2 + |\lambda_0|^2} \frac{\lambda^2 + \lambda_0^2}{\lambda^2 + \overline{\lambda_0}^2} \quad (1.3)$$

uniformly on compact subsets of Ω_α , where

$$h(\lambda, \lambda_0) = \frac{\lambda^2}{(\lambda^2 - |\lambda_0|^2)(\lambda^2 + |\lambda_0|^2)}.$$

Let $L_n(u)$ denote the upper envelope of $\mathcal{P}_{n, \alpha}$, i.e.,

$$L_n(u) := \sup\{|P_n(u)| : P_n \in \mathcal{P}_{n, \alpha}\}.$$

Theorem 1.2. *Let $\lambda = \lambda(u)$ be defined as above, $\lambda_0 = \lambda(u_0)$ and define the reproducing kernel $k_{\Omega_\alpha}(u, u_0)$ by*

$$k_{\Omega_\alpha}(u, u_0) = k_{\mathbb{H}_+}(\lambda, \lambda_0) := \frac{2\lambda\overline{\lambda_0}}{(\lambda + \overline{\lambda_0})^2}.$$

Then

$$L_n(u) \sim e^{ng_{\Omega_\alpha}(u, \infty)} k_{\Omega_\alpha}(u, u). \quad (1.4)$$

Finally, we would like to mention that the proofs will show that these results are universal in the following sense. In fact, one could instead of polynomials consider rational functions with fixed collection of poles $C = \{c_1, \dots, c_g\}$ outside of \mathbb{D} . The solution for the same problem for the class $\mathcal{F}_{n, \alpha}$ of rational functions with its only poles in C of order at most n is denoted by F_{n, u_0} . Let $B(u) = \prod b_{\Omega_\alpha}(u, c_k)$. The limit

$$\lim_{n \rightarrow \infty} B(u)^n F_{n, u_0}(u) = e^{i\phi} \frac{1}{2} \left(1 + \frac{h(\lambda, \lambda_0)}{h(\lambda_0, \lambda_0)} \right) \frac{\lambda^2 - |\lambda_0|^2}{\lambda^2 + |\lambda_0|^2} \frac{\lambda^2 + \lambda_0^2}{\lambda^2 + \overline{\lambda_0}^2},$$

for every choice of C . In particular, the upper envelope of this family, denoted by M_n , satisfies

$$M_n(u) \sim e^{n \sum g_{\Omega_\alpha}(u, c_k)} k_{\Omega_\alpha}(u, u).$$

2 Szegő -Widom asymptotics

In order to have uniqueness, we fix the normalization for the extremal polynomials and the complex Green's function

$$\lim_{u \rightarrow \infty} ub_{\Omega_\alpha}(u, \infty) > 0 \quad \text{and} \quad b_{\Omega_\alpha}(u_0, \infty)^n P_{n,u_0}(u_0) > 0.$$

Let \mathcal{P}_n be the set of all polynomials of degree at most n and

$$\mathcal{P}_{n,\alpha} = \{P \in \mathcal{P}_n : \|P\|_{A_\alpha} \leq 1\}. \quad (2.1)$$

Since $A_\alpha \subset \mathbb{T}$, the map

$$P_n(u) \mapsto P_n^*(u) := u^n \overline{P_n(1/\overline{u})}, \quad (2.2)$$

is an involution on $\mathcal{P}_{n,\alpha}$. This shows that for $L_n(\infty) := 1/\|T_n\|_{A_\alpha}$ we have

$$L_n(\infty) = L_n(0) \quad (2.3)$$

and there exists $\phi \in \mathbb{R}$ such that $P_{n,\infty} = e^{i\phi} P_{n,0}^*$. We will give a solution of (2.3) by reducing it to a problem which was already considered by Yuditskii [15]. Let $A_0 = \mathbb{R} \setminus (-1, 1)$ and $\Omega_0 = (\mathbb{C} \setminus \mathbb{R}) \cup (-1, 1)$. The map

$$u(z) = \frac{z - z_0}{z - \overline{z_0}}, \quad z_0 = i \tan(\alpha/2),$$

maps Ω_0 conformally onto Ω_α . By $z(u)$ we denote its inverse map. Henceforth, if we use z and u simultaneously we have in mind $z(u)$ and $u(z)$, respectively. Defining $z_\infty = z(\infty)$ it is obvious that $z_\infty = \overline{z_0}$. Note that $z(e^{i\alpha}) = -1$ and $z(e^{-i\alpha}) = 1$. To the polynomial $E_n(z) = (z - z_\infty)^n$, we associate the weighted norm

$$\|Q_n\|_{\Pi(E_n)} := \sup_{x \in A_0} \left| \frac{Q_n(x)}{E_n(x)} \right| \quad \text{for } Q_n \in \mathcal{P}_n. \quad (2.4)$$

Lemma 2.1. *Let $Q_{n,z_0} \in \mathcal{P}_n$ be the solution of the extremal problem*

$$|Q_{n,z_0}(z_0)| = \sup\{|Q_n(z_0)| : Q_n \in \mathcal{P}_n, \|Q_n\|_{\Pi(E_n)} \leq 1\}.$$

Then there exists $\phi \in \mathbb{R}$ such that

$$P_{n,0}(u) = e^{i\phi} \frac{Q_{n,z_0}(z)}{E_n(z)}. \quad (2.5)$$

Proof. We have

$$C \frac{z - z_l}{z - z_\infty} = u(z) - u(z_l), \quad C = \frac{z_0 - z_\infty}{z_l - z_\infty}.$$

Hence, the map

$$P_n(u) \mapsto Q_n(z) := E_n(z)P_n(u(z)),$$

maps \mathcal{P}_n bijectively onto itself. Moreover,

$$\|P_n\|_{A_\alpha} = \sup_{x \in A_0} \left| \frac{Q_n(x)}{E_n(x)} \right| = \|Q_n\|_{\Pi(E_n)}.$$

Therefore,

$$\begin{aligned} |P_{n,0}(0)| &= \sup\{|P_n(0)| : P_n \in \mathcal{P}_{n,\alpha}\} \\ &= \sup \left\{ \left| \frac{Q_n(z_0)}{E_n(z_0)} \right| : Q_n \in \mathcal{P}_n, \|Q_n\|_{\Pi(E_n)} \leq 1 \right\} = \left| \frac{Q_{n,z_0}(z_0)}{E_n(z_0)} \right| \end{aligned}$$

and (2.5) holds. \square

In [15] an explicit solution for this kind of problem is given. First, let us mention that in [11] it is shown that for fixed α there may be $N \in \mathbb{N}$ such that for $n < N$ the extremal polynomial is just z^n . This corresponds to a special case in [15]. Since we are only interested in asymptotics we assume that $n > N$. We recall the theorem in a way that is convenient for our purpose. Let $\omega(z, I; \Omega)$ denote the harmonic measure of the domain Ω .

Theorem 2.2 ([15]). *Let $E_n(z)$ be a polynomial with zeros $Z = \{\bar{z}_1, \dots, \bar{z}_n\} \subset \mathbb{C}_-$ and $z_0 = ir \in i\mathbb{R}_{>0}$. Then there exists a unique $0 < x_n < 1$ such that $I_n = [-x_n, x_n]$ satisfies*

$$\sum_{\bar{z}_l \in Z \cup \{\bar{z}_0\}} \omega(\bar{z}_l, I_n; \Omega_0 \setminus I_n) = 1. \quad (2.6)$$

Let $\Omega_n = \Omega_0 \setminus I_n$ and set

$$\mathcal{I}(z) = \prod_{\bar{z}_l \in Z \cup \{\bar{z}_0\}} b_{\Omega_n}(z, \bar{z}_l)$$

and

$$s_n(z) = \sqrt{\frac{z_0^2 - 1}{z_0^2 - x_n^2} \frac{z^2 - x_n^2}{z^2 - 1}}, \quad s_n(z_0) = 1.$$

The extremal polynomial Q_{n,z_0} is up to the unimodular factor $e^{i\phi}$ uniquely given by

$$Q_{n,z_0}(z) = e^{i\phi} E_n(z) \left(\frac{1 + s_n(z)}{2s_n(z)} \frac{1}{\mathcal{I}(z)} + \frac{1 - s_n(z)}{2s_n(z)} \frac{z - z_0}{z - \bar{z}_0} \frac{\overline{E_n(\bar{z})}}{E_n(z)} \mathcal{I}(z) \right).$$

In particular,

$$L_n(z_0) = |E_n(z_0)| \exp \left(\sum_{z \in Z \cup \{\bar{z}_0\}} g_{\Omega_n}(\bar{z}_l, z_0) \right).$$

Hence, by Theorem 2.2 the solution of the extremal problem is given by

$$Q_{n,z_0}(z) = E_n(z) \left(\frac{1 + s_n(z)}{2s_n(z)} b_n(z)^{-(n+1)} + \frac{1 - s_n(z)}{2s_n(z)} \frac{(z - z_0)^{n+1}}{(z - \bar{z}_0)^{n+1}} b_n(z)^{n+1} \right), \quad (2.7)$$

where $b_n(z) = b_{\Omega_n}(z, z_\infty)$. We also abbreviate $g_n(z) = g_{\Omega_n}(z, z_\infty)$, $g(z) = g_{\Omega_0}(z, z_\infty)$, $b(z) = b_{\Omega_0}(z, z_\infty)$ and $\omega_n(E) = \omega(z_\infty, E; \Omega_n)$. Note that (2.6) reads

$$\omega_n(I_n) = \frac{1}{n+1} \quad (2.8)$$

in this case. Our goal is to find the limit of $b_{\Omega_n}(u, \infty)^n P_{n,0}(u)$. Due to the conformal invariance of the Green's function, this is equivalent to finding the asymptotics of

$$f_n(z) = \frac{b(z)^n Q_{n,z_0}(z)}{E_n(z)}.$$

By the maximum principle and Montel's theorem, there exist subsequences n_j such that f_{n_j} converges to an analytic function f uniformly on compact subsets of Ω_0 . We will show that all subsequences have the same limit.

Lemma 2.3. *Let $I_n = [-x_n, x_n]$. Then $x_n \rightarrow 0$ as $n \rightarrow \infty$ and*

$$\lim_n s_n(z)^2 = \frac{z_0^2 - 1}{z_0^2} \frac{z^2}{z^2 - 1},$$

where the limit is uniformly on compact subsets of Ω_0 . Moreover,

$$\lim_n \frac{1 - s_n(z)}{2s_n(z)} \frac{(z - z_0)^{n+1}}{(z - \bar{z}_0)^{n+1}} b(z)^n b_n(z)^{n+1} = 0,$$

uniformly on compact subsets of $\Omega_0 \setminus \{0\}$.

Proof. By the maximum principle (see [8, 12 Ch. IV, Sec 2]) $\omega_n(I_n)$ is an increasing function of x_n . Since $\omega_n(I_n) \rightarrow 0$ as $x_n \rightarrow 0$, we obtain the first statement and the second statement is clear. Finally, we notice that on a compact subset of $\Omega_0 \setminus \{0\}$

$$\lim_n \frac{1 - s_n(z)}{2s_n(z)} = \frac{1 - s(z)}{2s(z)},$$

which is analytic there. Since

$$\left| \frac{(z - z_0)}{(z - \bar{z}_0)} b_n(z) b(z) \right| < 1 \quad \text{on } \Omega_n$$

we obtain the last statement. \square

Theorem 2.4. *The domain Ω_α has Szegő-Widom asymptotics. That is, there exists $\phi \in \mathbb{R}$ such that uniformly on compact subsets of Ω_α we have*

$$\lim_{n \rightarrow \infty} \overline{b_{\Omega_\alpha}(u^*, \infty)^n P_{n,\infty}(u^*)} = e^{i\phi} \frac{1 + s(z)}{2s(z)} \frac{b_{\Omega_0}(z, 0)}{b_{\Omega_0}(z, \bar{z}_0)}, \quad (2.9)$$

where

$$s(z) = \sqrt{\frac{z_0^2 - 1}{z_0^2} \frac{z^2}{z^2 - 1}}, \quad s(z_0) = 1.$$

Proof. Solving the Dirichlet problem for the harmonic function

$$h(z_1) = g_{\Omega_0}(z_1, z) - g_{\Omega_n}(z_1, z)$$

in Ω_n shows

$$h(z_1) = \int_{I_n} g_{\Omega_0}(z, x) \omega(z_1, dx; \Omega_n).$$

The symmetry of the Green's function with respect to the variables z and z_1 leads to

$$g_{\Omega_0}(z, z_1) - g_{\Omega_n}(z, z_1) = \int_{I_n} g_{\Omega_0}(z, x) \omega(z_1, dx; \Omega_n). \quad (2.10)$$

Therefore,

$$\log \left| \frac{b(z)^n}{b_n(z)^n} \right| = n(g_n(z) - g(z)) = - \int_{I_n} g_{\Omega_0}(z, x) n \omega_n(dx).$$

By (2.8), $\chi_{I_n} n\omega_n(dx)$ converges to the delta distribution and therefore

$$\lim_{n \rightarrow \infty} \log \left| \frac{b(z)^n}{b_n(z)^n} \right| = -g_{\Omega_0}(z, 0).$$

In the same way we see that $\lim_{n \rightarrow \infty} \log |b_{\Omega_n}(z, \bar{z}_0)| = -g_{\Omega_0}(z, \bar{z}_0)$. Therefore, in combination with Lemma 2.3 we obtain for the limit function f that

$$|f(z)| = \left| \frac{1 + s(z)}{2s(z)} \frac{b_{\Omega_0}(z, 0)}{b_{\Omega_0}(z, \bar{z}_0)} \right|,$$

and hence

$$f(z) = \frac{1 + s(z)}{2s(z)} \frac{b_{\Omega_0}(z, 0)}{b_{\Omega_0}(z, \bar{z}_0)}.$$

This shows

$$\lim_{n \rightarrow \infty} \frac{b(z)^n Q_{n, z_0}(z)}{E_n(z)} = e^{i\phi} \frac{1 + s(z)}{2s(z)} \frac{b_{\Omega_0}(z, 0)}{b_{\Omega_0}(z, \bar{z}_0)}. \quad (2.11)$$

Due to the symmetry of the domain with respect to the real line we have $\overline{b_{\Omega_\alpha}(\bar{u}, \infty)} = b_{\Omega_\alpha}(u, \infty)$. This and the conformal invariance of the Green's function leads to

$$\begin{aligned} b_{\Omega_\alpha}(u, \infty)^n P_{n, \infty}(u) &= b_{\Omega_\alpha}(u, \infty)^n e^{i\phi} u^n \overline{P_{n, 0}(1/\bar{u})} \\ &= e^{i\phi} b_{\Omega_\alpha}(u, 0)^n \overline{P_{n, 0}(u^*)} \\ &= e^{i\phi} \overline{b_{\Omega_\alpha}(u^*, \infty)^n P_{n, 0}(u^*)}. \end{aligned}$$

This concludes the proof. \square

As a corollary of Theorem 2.4, we obtain (1.1). Recall that

$$f_n \sim g_n \iff \lim_n \frac{f_n}{g_n} = 1$$

and

$$\text{Cap}(A_\alpha) := \lim_{u \rightarrow \infty} |u b_{\Omega_\alpha}(u, \infty)|.$$

Corollary 2.5. *Let T_n denote the Chebyshev polynomials of A_α . Then*

$$\|T_n\|_{A_\alpha} \sim \cot(\alpha/4) \text{Cap}(A_\alpha)^{n+1}.$$

Proof. Due to (2.3),

$$L_n(0) = \frac{1}{\|T_n\|_{A_\alpha}}.$$

Let $w : \Omega_0 \rightarrow \mathbb{C}_+$ be defined by $w(z) = \sqrt{\frac{z-1}{z+1}}$ and $w(z_0) = w_0 = e^{i(\pi-\alpha)/2}$. Since $w(0) = i$, we obtain

$$|b_{\Omega_0}(z_0, 0)| = |b_{\mathbb{C}_+}(w_0, i)| = \left| \frac{e^{i(\pi-\alpha)/2} - i}{e^{i(\pi-\alpha)/2} + i} \right| = \tan(\alpha/4).$$

Moreover,

$$\text{Cap}(A_\alpha) = \left| \lim_{u \rightarrow \infty} u b_{\Omega_\alpha}(u, \infty) \right| = \left| \lim_{u \rightarrow \infty} b_{\Omega_\alpha}(u, 0) \right| = |b_{\Omega_\alpha}(\infty, 0)|.$$

Thus, Theorem 2.4 shows that

$$\lim_{n \rightarrow \infty} \left| \frac{\text{Cap}(A_\alpha)^n}{\|T_n\|_{A_\alpha}} \right| = \frac{\tan(\alpha/4)}{\text{Cap}(A_\alpha)},$$

which concludes the proof. \square

The next natural question is to solve this problem not only for $u_0 = 0$, but for an arbitrary point $u_0 \in \Omega_\alpha$. As before, due to the symmetry of the domain, we can reduce it to $u_0 \in \mathbb{D}$. Namely, if $|u_0| > 1$, we have

$$P_{n,u_0} = P_{n,u_0^*}^*, \quad b_{\Omega_\alpha}(u, \infty)^n P_{n,u_0}(u) = \overline{b_{\Omega_\alpha}(u^*, \infty)^n P_{n,u_0^*}(u^*)}. \quad (2.12)$$

Lemma 2.6. *Let $|u_0| < 1$ and $z_{u_0} = z(u_0)$. Let K_0 be the unique circle that passes through z_{u_0} and \bar{z}_{u_0} such that Ω_0 is symmetric with respect to reflection by K_0 . Moreover, let $x_0 = K_0 \cap (-1, 1)$ and*

$$s(z, z_{u_0}) = \sqrt{\frac{z_{u_0}^2 - 1}{(z_{u_0} - x_0)^2} \frac{(z - x_0)^2}{z^2 - 1}}, \quad s(z_{u_0}, z_{u_0}) = 1.$$

Then there exists $\phi \in \mathbb{R}$ such that uniformly on compact subsets of Ω_α

$$\lim_{n \rightarrow \infty} b_{\Omega_\alpha}(u, \infty)^n P_{n,u_0}(u) = e^{i\phi} \frac{1 + s(z, z_{u_0})}{2s(z, z_{u_0})} \frac{b_{\Omega_0}(z, x_0)}{b_{\Omega_0}(z, \bar{z}_{u_0})}.$$

Proof. Let $u_0 \in \mathbb{D}$. By a Möbius transformation ψ , (aka Blaschke factor of the disc) we map $u_0 \mapsto 0$ such that A_α is mapped onto $A_{\alpha'}$ for some α' , i.e., $A_{\alpha'}$ is still symmetric with respect to the real axis and $1 \in A_{\alpha'}$. Then we compose this map with z (related to α') of the previous section in order to obtain a conformal map $\tilde{z} : \Omega_\alpha \rightarrow \Omega_0$ such that

$$\tilde{z}(e^{i\alpha}) = -1, \quad \tilde{z}(e^{-i\alpha}) = 1, \quad \tilde{z}(u_0) = i \tan(\alpha'/2).$$

Hence, we can apply exactly the same procedure in proving the asymptotics

$$\lim_j \frac{b(\tilde{z}, \tilde{z}_\infty; \Omega_0)^{n_j} \tilde{Q}_{n_j, \tilde{z}_0}(\tilde{z})}{\tilde{E}_{n_j}(\tilde{z})} = \frac{1 + \tilde{s}(\tilde{z})}{2\tilde{s}(\tilde{z})} \frac{b(\tilde{z}, 0)}{b(\tilde{z}, \tilde{z}(u_0))},$$

where

$$\tilde{s}(\tilde{z})^2 = \frac{\tilde{z}(u_0)^2 - 1}{\tilde{z}(u_0)^2} \frac{\tilde{z}^2}{\tilde{z}^2 - 1}.$$

The map $\phi : \Omega_0 \rightarrow \Omega_0$ with $\phi(\tilde{z}) = z$ is a fractional linear transformation (FLT) with $\phi(\mathbb{R}) = \mathbb{R}$.

$$\begin{array}{ccc} \overline{\mathbb{C}} \setminus A_\alpha & \xrightarrow{z} & \Omega_0 \\ \downarrow \psi & & \uparrow \phi \\ \overline{\mathbb{C}} \setminus A_\alpha & \xrightarrow{\tilde{z}} & \Omega_0 \end{array}$$

Due to properties of conformal maps in particular of FLTs we obtain $\phi(i\mathbb{R}) = K_0$, $\phi(0) = x_0$, $\phi(\tilde{z}(u_0)) = z_{u_0}$ and $\phi(\overline{\tilde{z}(u_0)}) = \overline{z_{u_0}}$, which concludes the proof. \square

Proof of Theorem 1.1. The function λ given by (1.2) is a composition of the maps $z : \Omega_\alpha \rightarrow \Omega_0$, $w : \Omega_0 \rightarrow \mathbb{C}_+$, defined by $w(z) = \sqrt{\frac{z-1}{z+1}}$ and $\tilde{\lambda}(w) : \mathbb{C}_+ \rightarrow \Pi$ defined by $\lambda(w) = \sqrt{-iw}$. Let $w_0 = w(z_{u_0})$. Using the reflection principle and that FLTs map circles onto circles, we obtain that $w(x_0) = i|w_0|$ and $w(\overline{z_{u_0}}) = -\overline{w_0}$. Hence,

$$\lim_{n \rightarrow \infty} b_{\Omega_\alpha}(u, \infty)^n P_{n, u_0}(u) = e^{i\phi} \frac{1}{2} \left(1 + \frac{v(w, w_0)}{v(w_0, w_0)} \right) \frac{w - i|w_0|}{w + i|w_0|} \frac{w + w_0}{w + \overline{w_0}},$$

where $v(w, w_0) = \frac{w}{(w+i|w_0|)(w-i|w_0|)}$. By definition $w = i\lambda^2$, which proves (1.3). \square

We define

$$L(u) := \lim_n e^{-ng_{\Omega_\alpha}(u, \infty)} L_n(u).$$

Note that (2.12) in particular implies that $L(u_0^*) = L(u_0)$.

Let us point out that the fact that we don't give a formula for P_{n, u_0} for $|u_0| = 1$ is just a consequence of our technique. Indeed, this question leads to a real Chebyshev problem, which was already introduced and solved by Chebyshev [2]. Later this problem was widely discussed; see e.g. [1, 7]. We only use that for each n there exists a maximizer P_{n, u_0} (either by referring to the real Chebyshev problem or by compactness of $\mathcal{P}_{n, \alpha}$). Due to Montel's theorem, $b_{\Omega_\alpha}(u, \infty)^n P_{n, u_0}(u)$ has a convergent subsequence, i.e., there exists $f(u)$ such that $f(u) = \lim_j b_{\Omega_\alpha}(u, \infty)^{n_j} P_{n_j, u_0}(u)$. Set $L(u_0) = f(u_0)$. We will see that with this definition $L(u)$ is continuous and therefore this value is independent of the particular choice of the subsequence.

Lemma 2.7. *$L_n(u)$ and $L(u)$ are continuous on $\mathbb{C} \setminus A_\alpha$ and Ω_α , respectively.*

Proof. Let $P \in P_{n, \alpha}$. Since $|b_{\Omega_\alpha}(u, \infty)^n P(u)| \leq 1$, P is locally bounded and therefore $\mathcal{P}_{n, \alpha}$ is equicontinuous. Hence, for every $u_0 \in \mathbb{C} \setminus A_\alpha$ there exists $\delta > 0$ such that $|u - u_0| < \delta$ implies

$$L_n(u_0) \geq |P_{n, u}(u_0)| > |P_{n, u}(u)| - \epsilon = L_n(u) - \epsilon.$$

In the same way we see that $L_n(u) > L_n(u_0) - \epsilon$ and therefore $|L_n(u) - L_n(u_0)| < \epsilon$. The same proof applies for $L(u)$. \square

3 Log Subharmonicity and Reproducing Kernels

In this chapter we will prove some properties of the extremal values $L_n(u)$ and $L(u)$ as functions on Ω_α . We recall the definition of log subharmonicity.

Let $\Omega \subset \mathbb{C}$ and $f : \Omega \rightarrow \mathbb{R}$ be an upper semicontinuous function. It is called subharmonic if for every $z_0 \in \Omega$ there exists R such that $\{z : |z - z_0| \leq R\} \subset \Omega$ and for all $0 < r \leq R$ we have

$$f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt;$$

cf. [4]. A function is called log subharmonic if $\log f$ is subharmonic.

Remark. If f is twice continuously differentiable, then f is subharmonic if and only if $\Delta f \geq 0$ in Ω .

Proposition 3.1. $L_n(u)$ and $L(u)$ are log subharmonic on $\mathbb{C} \setminus A_\alpha$ and Ω_α , respectively.

Proof. The modulus of an analytic function is log subharmonic. Since L_n is continuous it can be easily seen that it is log subharmonic as the upper envelope of polynomials; cf. [5, Lecture 7]. Clearly, this also holds for $|b_{\Omega_\alpha}(u, \infty)|^n L_n(u)$. Note that

$$\log |b_{\Omega_\alpha}(u, \infty)^n L_n(u)| = n(g_n(z) - g(z)) + g_{\Omega_n}(z, \bar{z}_{u_0})$$

By the maximum principle $g_{\Omega_n}(z, z_1)$ is increasing in n and therefore this holds for $n(g_n(z) - g(z)) + g_{\Omega_n}(z, \bar{z}_{u_0})$. Thus, we can interchange limit and integration and obtain that $L(u)$ is log subharmonic. \square

Proof of Theorem 1.2. Evaluating (1.3) at u_0 , we obtain

$$\begin{aligned} L(u_0) &= \left| \frac{\lambda_0^2 - |\lambda_0|^2}{\lambda_0^2 + |\lambda_0|^2} \right| \frac{2|\lambda_0|^2}{|\lambda_0^2 - \bar{\lambda}_0^2|} \\ &= \frac{|(\lambda_0^2 - |\lambda_0|^2)(\bar{\lambda}_0^2 + |\lambda_0|^2)|}{|\lambda_0^2 + |\lambda_0|^2|^2} \frac{2|\lambda_0|^2}{|\lambda_0^2 - \bar{\lambda}_0^2|} \\ &= \frac{|\lambda_0|^2 |\lambda_0^2 - \bar{\lambda}_0^2|}{|\lambda_0|^2 |\lambda_0 + \bar{\lambda}_0|^2} \frac{2|\lambda_0|^2}{|\lambda_0^2 - \bar{\lambda}_0^2|} \\ &= \frac{2|\lambda_0|^2}{|\lambda_0 + \bar{\lambda}_0|^2} = k_{\mathbb{H}^+}(\lambda_0, \lambda_0). \end{aligned}$$

\square

Remark. (i) Let $\partial, \bar{\partial}$ denote the Wirtinger derivatives. Since $4\partial\bar{\partial} = \Delta$, we have for twice continuously differentiable functions that

$$f \text{ is log subharmonic} \iff \begin{bmatrix} f(z) & \bar{\partial}f(z) \\ \partial f(z) & \partial\bar{\partial}f(z) \end{bmatrix} \geq 0.$$

Note that this matrix-inequality, which appears naturally for L as limit of an upper envelope of polynomials is just a small part of a matrix-inequality, which holds for reproducing kernels of analytic functions. Namely, writing

$$k(z, z_0) = \sum_k \phi_k(z) \overline{\phi_k(z_0)},$$

for an orthonormal basis $\{\phi_k\}$, we see that the matrix $\{\partial^i \bar{\partial}^j k(z_0, z_0)\}_{i,j=1}^n$ is the Gram matrix of the vectors $\{\phi_k(z_0)\}, \{\partial \phi_k(z_0)\}, \dots, \{\partial^n \phi_k(z_0)\}$ with respect to the standard ℓ^2 scalar product and therefore

$$\{\partial^i \bar{\partial}^j k(z_0, z_0)\}_{i,j=1}^n \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

- (ii) The kernel $\frac{1}{(\lambda + \lambda_0)^2}$ is up to normalization that Bergman kernel of $\mathbb{H}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Hence, $k_{\mathbb{H}_+}(\lambda, \lambda_0)$ is the reproducing kernel for the weighted Bergman space $\lambda A^2(\mathbb{H}_+)$.
- (iii) Since Π is a subset of \mathbb{H}_+ we assume that $k_{\Omega_\alpha}(u, u_0)$ has an extension to some larger domain (probably a Riemann surface). This is quite understandable for the following reason. In general, a reproducing kernel $k(z_0, z_0)$ diverges if z_0 converges to the boundary of the domain. But due to our setting, it is clear that $L(u) \rightarrow 1$ as $u \rightarrow A_\alpha$. Hence, A_α might not be real boundary of the domain on which $k_{\Omega_\alpha}(u, u_0)$ is defined.

References

- [1] N. I. Achieser, *Vorlesungen über Approximationstheorie*, Akademie-Verlag, Berlin, 1953.
- [2] P.L. Chebychev, *Sur les questions de minima qui se rattachent à la représentation approximative des fonctions*, Mém. Acad. Sci. Pétersb. **7** (1859), 199–291.
- [3] J. Christiansen, B. Simon, and M. Zinchenko, *Asymptotics of Chebychev Polynomials, I. Subsets of \mathbb{R}* , arXiv:1505.02604 (2015).
- [4] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970.
- [5] B. Y. Levin, *Lectures on entire functions*, Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence, RI, 1996, In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.
- [6] D. S. Lubinsky, *On the Bernstein constants of polynomial approximation*, Constr. Approx. **25** (2007), no. 3, 303–366.

- [7] A.A. Markov, *Lectures on functions deviating least from zero.*, (1906), 244–291, (Selected papers, Gostehizdat, Moscow- Leningrad, 1948, in Russian).
- [8] R. Nevanlinna, *Analytic functions*, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162, Springer-Verlag, New York-Berlin, 1970.
- [9] M. L. Sodin and P. M. Yuditskiĭ, *Functions that deviate least from zero on closed subsets of the real axis*, Algebra i Analiz **4** (1992), no. 2, 1–61.
- [10] G. Szegő, *Bemerkungen zu einer Arbeit von Herrn M. Fekete: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **21** (1924), no. 1, 203–208.
- [11] J.-P. Thiran and C. Dettaille, *Chebyshev polynomials on circular arcs in the complex plane*, Progress in approximation theory, Academic Press, Boston, MA, 1991, pp. 771–786.
- [12] V. Totik, *Chebyshev constants and the inheritance problem*, J. Approx. Theory **160** (2009), no. 1-2, 187–201.
- [13] ———, *Chebyshev polynomials on compact sets*, Potential Anal. **40** (2014), no. 4, 511–524.
- [14] H. Widom, *Extremal polynomials associated with a system of curves in the complex plane*, Advances in Math. **3** (1969), 127–232.
- [15] P. Yuditskii, *A complex extremal problem of Chebyshev type*, J. Anal. Math. **77** (1999), 207–235.